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# Some Results on General Class of Beta Integrals Involving Certain Particular Functions

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**ABSTRACT:** In this paper particular functions are used in general class of Beta integrals. Additionally, this class is used to assess specific integrals for some particular functions. The class of integrals studied in this paper are integrals involving the product of several exponential functions and Gauss's hypergeometric function. Several integrals for specific functions are obtained by applying this general integral to them.

**KEYWORDS:** Beta function, Gamma function, Exponential functions, Gauss's hypergeometric function, Extended Beta function, Generalized Hypergeometric function, Hurwitz-Lerch zeta function.

## I. INTRODUCTION

Choudhary et. al. [1][2] studied the extended form of Classical Euler Gamma and Beta function .They extended these functions to the entire complex plane by inserting the regularization factors $\exp\left(\frac{-A}{t}\right)$ and $\exp\left(\frac{-A}{t(1-t)}\right)$ in the integrands. They studied the following extended form of Gamma and Beta function respectively:

$$\Gamma_A(\alpha)=\int_0^\infty t^{\alpha-1}\exp\left(t-\frac{A}{t}\right)dt;\quad \mathrm{Re}(A)>0 \tag{1.1}$$

$$B(\alpha,\beta;A)=\int_0^\infty t^{\alpha-1}(1-t)^{\beta-1}\exp\left(\frac{-A}{t(1-t)}\right)dt;\quad \mathrm{Re}(A)>0 \tag{1.2}$$

If we take A=0 in equations (1.3) and (1.4) these functions reduce to their following original form respectively [11]

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$$\Gamma(\alpha)=\int_0^\infty t^{\alpha-1}e^{-t}dt \qquad [1.3]$$

$$B(\alpha,\beta)=\int_0^\infty t^{\alpha-1}(1-t)^{\beta-1}dt \;;\quad \mathrm{Re}(\alpha)>0,\ \mathrm{Re}(\beta)>0 \qquad [1.4]$$

These forms of Beta integral are further unified by the following general integrals. [4]

$$I_{\alpha,\beta}\left[f(t)\right]=\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}f(t)dt;\quad \mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0 \qquad (1.5)$$

And [6, p.1995, eq.(2.1)]

$$I_{A,\alpha,\beta,\gamma;a,b}\left[\phi(t),\psi(t)\right]=\frac{1}{B(\alpha,\beta)}\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\left[\phi(t)\right]^{\gamma}\exp\left[-A\psi(t)\right]dt;$$

(1.6)

$$\{\mathrm{Re}(\alpha)>0,\mathrm{Re}(\beta)>0,\mathrm{Re}(A)\}>0$$

Where $\phi(t)$, $\psi(t)$ and $f(t)$ are particular functions and $\phi(t)\neq 0,\ \ a<t<b,\ \ a\neq b$

Inspired by specific series integral representations of the Hurwitz-Zeta function $\zeta(z,a)$ and Hurwitz Lerch zeta function $\phi(z,s,a)$ ([7][8]), Jaimini and Sharma [5] recently investigated the extended beta function $B(\alpha,\beta;A)$ using the subsequent general series integral representation.

$$B(\alpha,\beta;A)=\sum_{k=0}^{\infty}\frac{(\lambda)_k}{k!}\int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\exp\left[-(A+kp)\psi(t)\right].\left[1-\exp\left(-p\psi(t)\right)\right]^{\lambda}$$

(1.7)

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Where $\psi(t) = \dfrac{1}{t(1-t)}$

On summing the series in (1.7) and using binomial expansion it reduce to the original definition in (1.2).

## II. DEFINITIONS AND RESULTS REQUIRED

The generalized hypergeometric function is defined and represented by the following series [11]

$$ {}_pF_q\left[\begin{matrix} a_1,\dots,a_p; \\ \\ b_1,\dots,b_q; \end{matrix} \quad z\right] = \sum_{r=0}^{\infty} \frac{\prod_{i=1}^{p}(a_i)_n}{\prod_{j=1}^{q}(b_j)_n n!}, \qquad (2.1) $$

$$ p,q \in z^+; \quad b_j \neq 0,-1,-2,\dots; \quad i=1,2,\dots,r $$

The exponential function ${}_0F_0[-,-;z]$ and the Gauss's hypergeometric function ${}_2F_1[a,b;c;z]$ are special cases of the above hypergeometric function ${}_pF_q[\cdot]$

It is useful here to define the following relations for Pocchammer symbol [11, pp.21-23; eqs. (4), (15), (18),(20),(24)]

$$ (a)_n = \begin{bmatrix} a(a+1)\dots(a+n-1); & if & n=1,2,\dots\dots\dots \\ 1; & if & n=0 \end{bmatrix} $$

$$ = \frac{\Gamma(a+n)}{\Gamma(a)} \qquad (2.2) $$

$$ \Gamma(a-n) = \frac{(-1)^n \Gamma(a)}{(1-a)_n} \qquad (2.3) $$

$$ (\lambda)_{m+n} = (\lambda)_m (\lambda+m)_n \qquad (2.4) $$

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The duplication formula defined in the following form.

$$(\lambda)_{2n} = 2^{2n}\left(\frac{\lambda}{2}\right)_n\left(\frac{\lambda+1}{2}\right)_n \tag{2.5}$$

The following integrals are required in the sequel [9, pp.224-225]

$$\int_a^b \frac{(t-a)^{\zeta}(b-t)^{\beta-1}}{[\phi(t)]^{\zeta+\beta+1}}\,{}_2F_1\left[\begin{matrix}\xi,\eta\\ \beta\end{matrix};\frac{(1+\mu)(b-t)}{\phi(t)}\right]dt$$

$$=\frac{(b-a)^{-1}(1+\lambda)^{-\zeta-1}(1-\mu)^{-\beta}\,\Gamma\beta\,\Gamma(\zeta+1)\,\Gamma(\zeta+\beta-\xi+1)}{\Gamma(\zeta+\beta-\xi+1)\,\Gamma(\zeta+\beta-\eta+1)}$$

$$\mathrm{Re}(\zeta)>-1,\quad \mathrm{Re}(3-\xi-\eta-2\zeta)>0, \tag{2.6}$$

$$\int_a^b \frac{(t-a)^{\zeta}(b-t)^{\zeta}}{[\phi(t)]^{2\zeta+2}}\,{}_2F_1\left[\begin{matrix}\xi,\eta\\ \frac{(\xi+\eta+1)}{2}\end{matrix};\frac{(1+\mu)(b-t)}{\phi(t)}\right]dt$$

$$=\frac{\pi\,\Gamma(\zeta+1)\,\Gamma\frac{(\xi+\eta+1)}{2}\Gamma\left(\zeta+\frac{(3-\xi-\eta)}{2}\right)}{2^{2\zeta+2}(b-a)\left[(1+\lambda)(1+\mu)^{\zeta+1}\right]\Gamma\left(\frac{\xi+1}{2}\right)\ \Gamma\left(\frac{\eta+1}{2}\right)\Gamma\left(\zeta+\frac{3-\xi}{2}\right)\Gamma\left(\zeta+\frac{3-\eta}{2}\right)}$$

$$\mathrm{Re}(\zeta)>-1,\quad \mathrm{Re}(3-\xi-\eta+2\zeta)>0, \tag{2.7}$$

$$\int_a^b \frac{(t-a)^{\beta-\zeta}(b-t)^{\beta-1}}{[\phi(t)]^{2\beta-\zeta+1}}\,{}_2F_1\left[\begin{matrix}\xi,1-\xi\\ \xi\end{matrix};\frac{(1+\mu)(b-t)}{\phi(t)}\right]dt$$

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$$= \frac{\pi\,\Gamma(\zeta)\Gamma(\beta)\Gamma(\beta-\zeta+1)}{2^{2\beta-1}(b-a)(1+\lambda)^{1+\beta-\zeta}(1+\mu)^{\beta}\Gamma\left(\frac{1-\xi+\zeta}{2}\right)\Gamma\left(\frac{\xi+\zeta}{2}\right)}$$

$$\frac{1}{\Gamma\left(\beta+\frac{\xi-\zeta+1}{2}\right)\Gamma\left(\beta+\frac{2-\xi-\zeta}{2}\right)}$$

$$\text{Re}(\beta)>0, \text{Re}(\beta-\zeta+1)>0 \qquad (2.8)$$

## III. MAIN RESULTS

In this section we will study the following general class of integrals in view of the series integral representation of extended Beta function in (1.7)

(i) $$I_{A,\alpha,\beta,\gamma;\lambda_1,p_1,\ldots,\lambda_r,p_r;\xi,\eta,c,d;a,b}\left[\phi(t),\psi(t),f(t)\right]$$

$$=\frac{1}{B(\alpha,\beta)}\int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\left[\phi(t)\right]^{\gamma}\exp\left[-A\psi(t)\right]$$

$$\prod_{i=1}^{r}\left[1-\exp\left(-p_i\psi(t)\right)\right]^{\lambda_i}\cdot {}_2F_1\left[\begin{matrix}\xi,\eta\\ c\end{matrix};df(t)\right]dt,$$

(3.1)

Where $\phi(t),\psi(t)$ and $f(t)$ are some particular function

$\text{Re}(\alpha), \text{Re}(\beta), \text{Re}(A)>0$, $\lambda_i$, $p_i>0$ for $i=1,2,\ldots,r$

**EVALUATION OF INTEGRALS: -**

The following three integrals are evaluated for certain values of the particular functions $\phi(t),\psi(t)$ and $f(t)$ in our main general class of integrals considered in (3.1).

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**Integral-1**

For $\psi(t)=\frac{[\phi(t)]}{(t-a)}, f(t)=\frac{(1+\mu)(b-t)}{[\phi(t)]}$

and $\gamma=-(\alpha+\beta)$, $c=\beta$, $d=1$

We have

$$I_{A,\alpha,\beta,-(\alpha+\beta),\lambda_1,p_1,\ldots,\lambda_r,p_r,\xi,\eta,\beta,1;a,b}\left[\phi(t),\frac{\phi(t)}{(t-a)},\frac{(1+\mu)(b-t)}{\phi(t)}\right]$$

$$=\frac{(b-a)^{-1}(1+\lambda)^{-\alpha}(1-\mu)^{\beta}\Gamma(\alpha+\beta)\Gamma(\alpha+\beta-\xi-\eta)}{\Gamma(\alpha+\beta-\xi)\Gamma(\alpha+\beta-\eta)}.$$

$$\sum_{n_1,\ldots n_r=0}^{\infty}\frac{\prod_{i=1}^{r}\left[(-\lambda_i)_{n_i}\right]}{n_1!\ldots n_r!}\,{}_2F_2\left[\begin{matrix}(1-\alpha-\beta+\xi), & (1-\alpha-\beta+\eta)\\ (1-\alpha)\ , & (1-\alpha-\beta+\xi+\eta)\end{matrix};-\left(A+\sum_{i=1}^{r}n_i p_i\right)(1+\lambda)\right]$$

(3.2)

Where $\phi(t)=(b-a)+\lambda(t-a)+\mu(b-t)\neq 0,$

$\mathrm{Re}(\alpha),\mathrm{Re}(\beta),\mathrm{Re}(\alpha+\beta-\xi-\eta)>0$ and the result in (3.2) exists.

**Integral-2**

For

$$\psi(t)=[\phi(t)]^2[(t-a)(b-t)]^{-1}$$

$$f(t)=(1+\mu)(b-t)[\phi(t)]^{-1}$$

and $\alpha=\beta=\zeta+1,$, $\gamma=-(2\zeta+2)$, $c=\frac{\xi+\eta+1}{2}$, $d=1$

We have

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$$I_{A,\zeta+1,\zeta+1,-(2\zeta+2);\lambda_1,p_1\ldots,\lambda_r,p_r,\xi,\eta,\frac{\xi+\eta+1}{2},1;a,b}$$

$$\left[\phi(t),[\phi(t)]^2\left[(t-a)(b-t)\right]^{-1},(1+\mu)(b-t)[\phi(t)]^{-1}\right].$$

$$=\frac{\pi\,\Gamma\left(\frac{\xi+\eta+1}{2}\right)\Gamma(\zeta+1)\Gamma\left(\zeta+\frac{3-\xi-\eta}{2}\right)}{2^{2\zeta+1}(b-a)\left[(1+\lambda)(1+\mu)\right]^{\zeta+1}\Gamma\left(\frac{\xi+1}{2}\right)\Gamma\left(\frac{\eta+1}{2}\right)\Gamma\left(\zeta+\frac{3-\xi}{2}\right)\Gamma\left(\zeta+\frac{3-\eta}{2}\right)}$$

$$\sum_{n_1,\ldots n_r=0}^{\infty}\frac{\prod_{i=1}^{r}\left[(-\lambda_i)_{n_i}\right]}{n_1!\ldots n_r!}\,{}_2F_2\left[\begin{matrix}\left(\frac{\xi-1}{2}-\zeta\right), & \left(\frac{\eta-1}{2}-\xi\right)\\ -\zeta, & \left(\frac{\xi+\eta+1}{2}-\zeta\right)\end{matrix};\,-4\left(A+\sum_{i=1}^{r}n_ip_i\right)(1+\lambda)(1+\mu)\right] \tag{3.3}$$

Where $\phi(t)=(b-a)+\lambda(t-a)+\mu(b-t)\neq 0,\ a\neq b,\ \lambda\neq -1,\ \mathrm{Re}(\xi)>-1$ and the result in (3.3) exists.

**Integral-3**

For $\psi(t)=[\phi(t)]^2\left[(t-a)(b-t)\right]^{-1}, f(t)=(1+\mu)(b-t)[\phi(t)]^{-1}$

and $\alpha=\beta-v+1,,\ \gamma=-(2\beta-v+1),$

$\eta=1-\xi, c=v, d=1$

We have

$$I_{A,\beta-v+1,\beta,-(2\beta-v+1),\lambda_1,p_1\ldots,\lambda_r,p_r,-\xi,1-\xi,v,1;a,b}.$$

$$\left[\phi(t),[\phi(t)]^2\left[(t-a)(b-t)\right]^{-1},(1+\mu)(b-t)[\phi(t)]^{-1}\right]$$

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$$= \frac{\pi\, \Gamma(v)\Gamma(2\beta - v + 1)(b-a)^{-1}(1+\lambda)^{v-1-\beta}(1+\mu)^{-\beta}}{2^{2\beta-1}\Gamma\left(\frac{1-\xi+v}{2}\right)\Gamma\left(\frac{\xi+v}{2}\right)\Gamma\left(\beta+\frac{\xi-v+1}{2}\right)\Gamma\left(\beta+\frac{2-\xi-v}{2}\right)}$$

$$\sum_{n_1,\ldots,n_r=0}^{\infty} \frac{\prod_{i=1}^{r}\left[(-\lambda_i)_{n_i}\right]}{n_1!\ldots n_r!}\, {}_2F_2\left[\begin{matrix}\left(-\beta+\frac{1-\xi+v}{2}\right),\left(-\beta+\frac{\xi+v}{2}\right) & ; & \frac{-4\left(A+\sum_{i=1}^{r} n_i p_i\right)}{(1+\lambda)^{-1}(1+\mu)^{-1}} \\ (1-\beta) \quad ,(v-\beta) & ; & \end{matrix}\right]$$

(3.4)

Where $\phi(t) = (b-a) + \lambda(t-a) + \mu(b-t) \neq 0,\ a \neq b, \lambda \neq -1$ and the result in (3.4) exists.

**OUT LINE OF PROOFS:**

**Proof of (3.2):**

To prove the result in (3.2) we denote its left-hand side by $\Delta_4$ i.e.

$$\Delta_4 = I_{A,\alpha,\beta,-(\alpha+\beta),\lambda_1,p_1,\ldots\lambda_r,p_r,\xi,\eta,\beta,1;a,b}\left[\phi(t), \frac{\phi(t)}{(t-a)}, \frac{(1+\mu)(b-t)}{\phi(t)}\right]$$

Now in view of the general notation of class of integrals (3.1) we have the following form of integral.

$$\Delta_4 = \frac{1}{B(\alpha,\beta)} \int_a^b (t-a)^{\alpha-1}(b-t)^{\beta-1}\left[\phi(t)\right]^{-(\alpha+\beta)} \exp\left[\frac{-A\phi(t)}{(t-a)}\right].$$

$$\prod_{i=1}^{r}\left[1-\exp\left(\frac{-p_i\phi(t)}{(t-a)}\right)\right]^{\lambda_i} {}_2F_1\left[\begin{matrix}\xi, \quad \eta \\ \beta\end{matrix} ; \frac{(1+\mu)(b-t)}{\phi(t)}\right] dt$$

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Now using binomial expansion and then expressing exponential function in series form ${}_0F_0$ with the help of (2.1)and then changing the order of summation and integration, we have,

$$\Delta_4 = \frac{1}{B(\alpha,\beta)} \sum_{m,n_1,\ldots,n_r=0}^{\infty} \frac{\prod_{i=1}^{r}\left[(-\lambda_i)_{n_i}\right]\left(-A-\sum_{i=1}^{r} n_i p_i\right)^m}{n_1!\ldots n_r! \quad m!}.$$

$$\int_a^b (t-a)^{\alpha-m-1}(b-t)^{\beta-1}\left[\phi(t)\right]^{-(\alpha+\beta)+m} {}_2F_1\left[\begin{matrix}\xi,\eta\\ \beta\end{matrix}; \frac{(1+\mu)(b-t)}{\phi(t)}\right]dt$$

Now on evaluating the inner integral with the help of (2.6) and then interpreting the $m$ series into ${}_2F_2$ in view of (2.1) we at once arrived at the desired result in (3.2).[3,4,5]

The results in (3.3) and (3.4) are proved following on similar lines as to prove the result (3.2) and using (2.7), (2.8) therein respectively[6,7,8]

**SPECIAL CASES:**

If in (3.2), (3.3) and (3.4) we take $r=1$ these reduce to the known results due to Pathan, Jaimini and Sharma [10, pp.5-8, Eqs.3.6-3.8] respectively.

If in the integrals (3.2), (3.3) and (3.4) we take $\lambda=0$ and $r=1$ then these result reduce to the known results due to Gautam [3,pp.284-285, eqs (6.3.6), (6.3.9), (6.3.8)[9,10,11]

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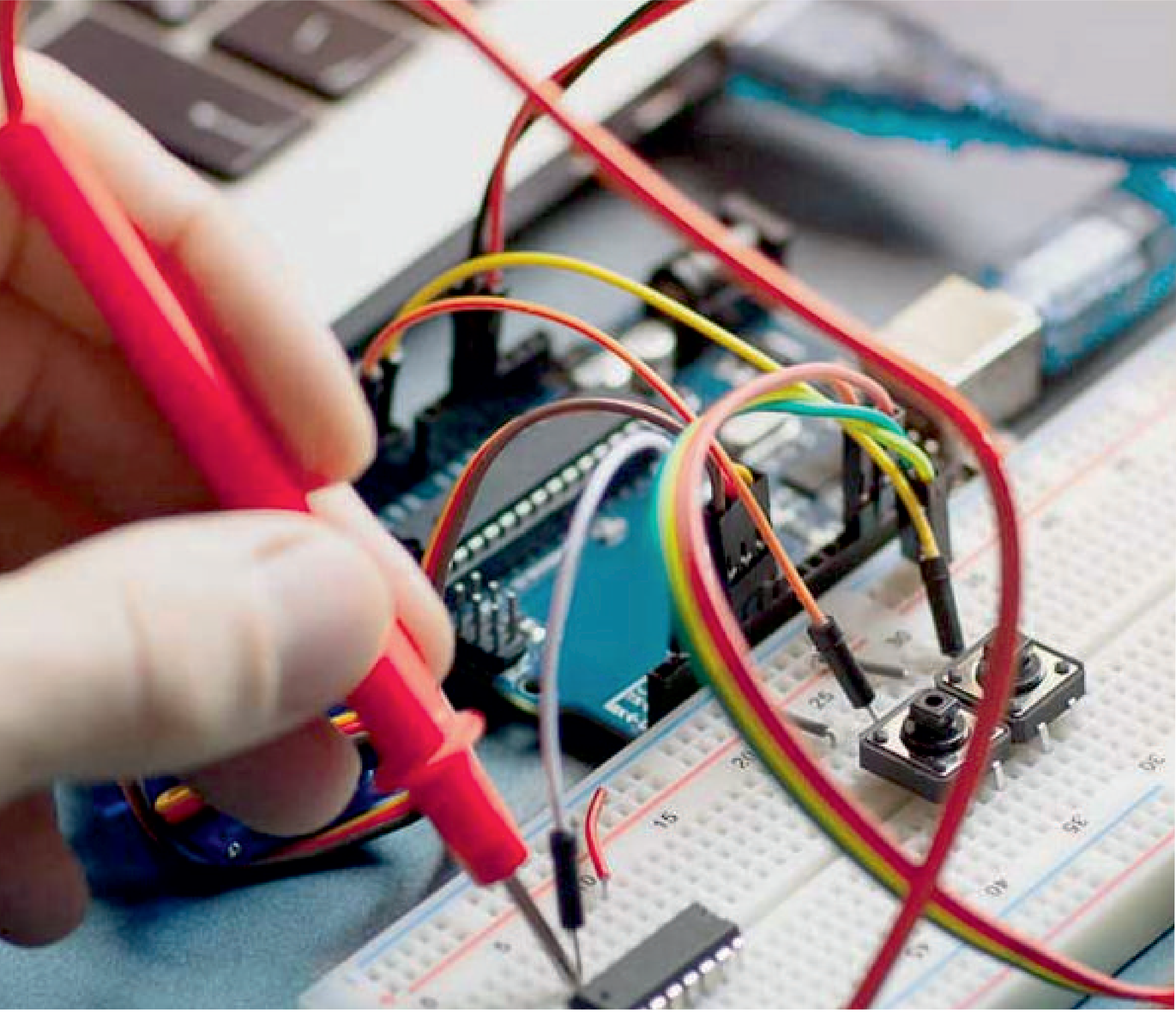


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