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Special Functions in Mathematics

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ABSTRACT: Special functions are particular mathematical functions that have more or less established names and notations due to their importance in mathematical analysis, functional analysis, geometry, physics, or other applications.

The term is defined by consensus, and thus lacks a general formal definition, but the list of mathematical functions contains functions that are commonly accepted as special.

KEYWORDS-special functions, mathematics, geometry, consensus, analysis

I. INTRODUCTION

Many special functions appear as solutions of differential equations or integrals of elementary functions. Therefore, tables of integrals[1] usually include descriptions of special functions, and tables of special functions[2] include most important integrals; at least, the integral representation of special functions. Because symmetries of differential equations are essential to both physics and mathematics, the theory of special functions is closely related to the theory of Lie groups and Lie algebras, as well as certain topics in mathematical physics.

Symbolic computation engines usually recognize the majority of special functions.

Most special functions are considered as a function of a complex variable. They are analytic; the singularities and cuts are described; the differential and integral representations are known and the expansion to the Taylor series or asymptotic series are available. In addition, sometimes there exist relations with other special functions; a complicated special function can be expressed in terms of simpler functions. Various representations can be used for the evaluation; the simplest way to evaluate a function is to expand it into a Taylor series. However, such representation may converge slowly or not at all. In algorithmic languages, rational approximations are typically used, although they may behave badly in the case of complex argument(s).[1,2,3]

History of special functions

Classical theory

While trigonometry and exponential functions were systematized and unified by the eighteenth century, the search for a complete and unified theory of special functions has continued since the nineteenth century. The high point of special function theory in 1800–1900 was the theory of elliptic functions; treatises that were essentially complete, such as that of Tannery and Molk,[3] expounded all the basic identities of the theory using techniques from analytic function theory (based on complex analysis). The end of the century also saw a very detailed discussion of spherical harmonics.

Changing and fixed motivations

While pure mathematicians sought a broad theory deriving as many as possible of the known special functions from a single principle, for a long time the special functions were the province of applied mathematics. Applications to the physical sciences and engineering determined the relative importance of functions. Before electronic computation, the importance of a special function was affirmed by the laborious computation of extended tables of values for ready look-up, as for the familiar logarithm tables. (Babbage's difference engine was an attempt to compute such tables.) For this purpose, the main techniques are:

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- numerical analysis, the discovery of infinite series or other analytical expressions allowing rapid calculation; and
- reduction of as many functions as possible to the given function.

More theoretical questions include: asymptotic analysis; analytic continuation and monodromy in the complex plane; and symmetry principles and other structural equations.[4,5,6]

Twentieth century

The twentieth century saw several waves of interest in special function theory. The classic Whittaker and Watson (1902) textbook[4] sought to unify the theory using complex analysis; the G. N. Watson tome A Treatise on the Theory of Bessel Functions pushed the techniques as far as possible for one important type, including asymptotic results.

The later Bateman Manuscript Project, under the editorship of Arthur Erdélyi, attempted to be encyclopedic, and came around the time when electronic computation was coming to the fore and tabulation ceased to be the main issue.

Contemporary theories

The modern theory of orthogonal polynomials is of a definite but limited scope. Hypergeometric series, observed by Felix Klein to be important in astronomy and mathematical physics,[5] became an intricate theory, in need of later conceptual arrangement. Lie groups, and in particular their representation theory, explain what a spherical function can be in general; from 1950 onwards substantial parts of classical theory could be recast in terms of Lie groups. Further, work on algebraic combinatorics also revived interest in older parts of the theory. Conjectures of Ian G. Macdonald helped to open up large and active new fields with the typical special function flavour. Difference equations have begun to take their place besides differential equations as a source for special functions.

Special functions in number theory

In number theory, certain special functions have traditionally been studied, such as particular Dirichlet series and modular forms. Almost all aspects of special function theory are reflected there, as well as some new ones, such as came out of the monstrous moonshine theory.[7,8,9]

Special functions of matrix arguments

Analogues of several special functions have been defined on the space of positive definite matrices, among them the power function which goes back to Atle Selberg,[6] the multivariate gamma function,[7] and types of Bessel functions.[8]

The NIST Digital Library of Mathematical Functions has a section covering several special functions of matrix arguments.[9]

II. DISCUSSION

Common special functions :

- Bessel functions: solutions to differential equations and popular in problems involving circular or cylindrical symmetry or wave propagation.
- Beta functions: definite integrals, related to the gamma function.
- Chi Function: a special case of the Lerch transcendent; Resembles the Dirichlet series for the polylogarithm.
- Error Function: Important in the study of errors. It
- Gamma functions: the multivariate form is used extensively in many sub-fields of multivariate analysis, including Bayesian Psychometric Modeling and Signal Processing,
- Legendre functions, which are solutions to Legendre equations— used extensively in physics.

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- Trigonometric functions: functions of angles,
- Hypergeometric functions: building blocks for many other functions, including closed form solutions of linear differential equations with polynomial coefficients.

Perhaps surprisingly, special functions do not include those you'll come across in the first semester or two of calculus, like power functions, logarithm function, and exponential functions. Although they are certainly "classical" functions that have been around for centuries, they aren't very useful for solving physical problems. For that, you need more complex functions. Some authors do include the trig functions (cosine function, sine function, etc.) in the definition of special functions as a group (as opposed to individual functions, which are deemed as too simplistic on their own).[10,11,12]

III. RESULTS

According to legend, Leo Szilard's baths were ruined by his conversion to biology. He had enjoyed soaking for hours while thinking about physics. But as a convert he found this pleasure punctuated by the frequent need to leap out and search for a fact. In physics—particularly theoretical physics—we can get by with a few basic principles without knowing many facts; that is why the subject attracts those of us cursed with poor memory.

But there is a corpus of mathematical information that we do need. Much of this consists of formulas for the "special" functions. How many of us remember the expansion of cos 5x in terms of cos x and sin x, or whether an integral obtained in the course of a calculation can be identified as one of the many representations of a Bessel function, or whether the asymptotic expression for the gamma function involves (n + 1/2) or (n – 1/2)? For such knowledge, we theorists have traditionally relied on compilations of formulas. When I started research, my peers were using Jahnke and Emde's Tables of Functions with Formulae and Curves (J&E) [1] or Erdélyi and coauthors' Higher Transcendental Functions. [2]

Then in 1964 came Abramowitz and Stegun's Handbook of Mathematical Functions (A&S), [3] perhaps the most successful work of mathematical reference ever published. It has been on the desk of every theoretical physicist. Over the years, I have worn out three copies. Several years ago, I was invited to contemplate being marooned on the proverbial desert island. What book would I most wish to have there, in addition to the Bible and the complete works of Shakespeare? My immediate answer was: A&S. If I could substitute for the Bible, I would choose Gradsteyn and Ryzhik's Table of Integrals, Series and Products. [4] Compounding the impiety, I would give up Shakespeare in favor of Prudnikov, Brychkov and Marichev's of Integrals and Series. [5] On the island, there would be much time to think about physics and much physics to think about: waves on the water that carve ridges on the sand beneath and focus sunlight there; shapes of clouds; subtle tints in the sky. … With the arrogance that keeps us theorists going, I harbor the delusion that it would be not too difficult to guess the underlying physics and formulate the governing equations. It is when contemplating how to solve these equations—to convert formulations into explanations—that humility sets in. Then, compendia of formulas become indispensable.[13,14]

Nowadays the emphasis is shifting away from books towards computers. With a few keystrokes, the expansion of cos 5x, the numerical values of Bessel functions, and many analytical integrals can all be obtained easily using software such as Mathematica and Maple. (In the spirit of the times, I must be even handed and refer to both the competing religions.) A variety of resources is available online. The most ambitious initiative in this direction is being prepared by NIST, the descendant of the US National Bureau of Standards, which published A&S. NIST's forthcoming Digital Library of Mathematical Functions (DLMF) will be a free Web-based collection of formulas (dlmf.nist.gov), cross-linked and with live graphics that can be magnified and rotated. (Stripped-down versions of the project will be issued as a book and a CD-ROM for people who prefer those media.)

The DLMF will reflect a substantial increase in our knowledge of special functions since 1964, and will also include new families of functions. Some of these functions were (with one class of exceptions) known to mathematicians in 1964, but they were not well known to scientists, and had rarely been applied in physics. They are new in the sense that, in the years since 1964, they have been found useful in several branches of physics. For example, string theory and quantum chaology now make use of automorphic functions and zeta functions; in the theory of solitons and integrable dynamical systems, Painlevé transcendents are widely employed; and in optics and quantum mechanics, a central role

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is played by "diffraction catastrophe" integrals, generated by the polynomials of singularity theory—my own favorite, and the subject of a chapter I am writing with Christopher Howls for the DLMF.

This continuing and indeed increasing reliance on special functions is a surprising development in the sociology of our profession. One of the principal applications of these functions was in the compact expression of approximations to physical problems for which explicit analytical solutions could not be found. But since the 1960s, when scientific computing became widespread, direct and "exact" numerical solution of the equations of physics has become available in many cases. It was often claimed that this would make the special functions redundant. Similar skepticism came from some pure mathematicians, whose ignorance about special functions, and lack of interest in them, was almost total. I remember that when singularity theory was being applied to optics in the 1970s, and I was seeking a graduate student to pursue these investigations, a mathematician recommended somebody as being very bright, very knowledgeable, and interested in applications. But this student had never heard of Bessel functions (nor could he carry out the simplest integrations, but that is another story).

The persistence of special functions is puzzling as well as surprising. What are they, other than just names for mathematical objects that are useful only in situations of contrived simplicity? Why are we so pleased when a complicated calculation "comes out" as a Bessel function, or a Laguerre polynomial? What determines which functions are "special"? These are slippery and subtle questions to which I do not have clear answers. Instead, I offer the following observations.

There are mathematical theories in which some classes of special functions appear naturally. A familiar classification is by increasing complexity, starting with polynomials and algebraic functions and progressing through the "elementary" or "lower" transcendental functions (logarithms, exponentials, sines and cosines, and so on) to the "higher" transcendental functions (Bessel, parabolic cylinder, and so on). Functions of hypergeometric type can be ordered by the behavior of singular points of the differential equations representing them, or by a group-theoretical analysis of their symmetries. But all these classifications are incomplete, in the sense of omitting whole classes that we find useful. For example, Mathieu functions fall outside the hypergeometric class, and gamma and zeta functions are not the solutions of simple differential equations. Moreover, even when the classifications do apply, the connections they provide often appear remote and unhelpful in our applications.

One reason for the continuing popularity of special functions could be that they enshrine sets of recognizable and communicable patterns and so constitute a common currency. Compilations like A&S and the DLMF assist the process of standardization, much as a dictionary enshrines the words in common use at a given time. Formal grammar, while interesting for its own sake, is rarely useful to those who use natural language to communicate. Arguing by analogy, I wonder if that is why the formal classifications of special functions have not proved very useful in applications.[15]

Sometimes the patterns embodying special functions are conjured up in the form of pictures. I wonder how useful sines and cosines would be without the images, which we all share, of how they oscillate. In 1960, the publication in J&E of a three-dimensional graph showing the poles of the gamma function in the complex plane acquired an almost iconic status. With the more sophisticated graphics available now, the far more complicated behavior of functions of several variables can be explored in a variety of two-dimensional sections and three-dimensional plots, generating a large class of new and shared insights.

"New" is important here. Just as new words come into the language, so the set of special functions increases. The increase is driven by more sophisticated applications, and by new technology that enables more functions to be depicted in forms that can be readily assimilated.

IV. CONCLUSION

Sometimes the patterns are associated with the asymptotic behavior of the functions, or of their singularities. Of the two Airy functions, Ai is the one that decays towards infinity, while Bi grows; the J Bessel functions are regular at the origin, the Y Bessel functions have a pole or a branch point.

Perhaps standardization is simply a matter of establishing uniformity of definition and notation. Although simple, this is far from trivial. To emphasize the importance of notation, Robert Dingle in his graduate lectures in theoretical physics at the University of St. Andrews in Scotland would occasionally replace the letters representing variables by nameless invented squiggles, thereby inducing instant incomprehensibility. Extending this one level higher, to the

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names of functions, just imagine how much confusion the physicist John Doe would cause if he insisted on replacing sin x by doe(x), even with a definition helpfully provided at the start of each paper.

To paraphrase an aphorism attributed to the biochemist Albert Szent-Györgyi, perhaps special functions provide an economical and shared culture analogous to books: places to keep our knowledge in, so that we can use our heads for better things.[16]

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